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Relativistic extension of shape-invariant potentials

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Abstract
The Dirac equation for a charged spinor in an electromagnetic field is written for special cases of spherically symmetric potentials. This facilitates the introduction of relativistic extensions of shape-invariant potential classes. We obtain the relativistic spectra and spinor wavefunctions for all potentials in one of these classes. The nonrelativistic limit reproduces the usual Rosen–Mörse I and II, Eckart, Pöschl–Teller and Scarf potentials.

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Exactly solvable potentials in nonrelativistic quantum mechanics fall within distinct classes of 'shape-invariant potentials' [1–8]. Each potential in a given class can be mapped into another in the same class by a canonical transformation of the coordinates [7, 9–13]. The transformation gives a correspondence map among the potential parameters, angular momentum and energy. Using the resulting parameter substitution and the bound-state spectrum of the original potential one can easily and directly obtain the spectra of all other potentials in the class. Moreover, the eigenstate wavefunctions are obtained by simple transformations of those of the original potential. It is very tempting to search for the relativistic extensions of these classes and obtain the relativistic spectra of the bound states and corresponding spinor wavefunctions. In fact, one such class has already been established. Recently, the Dirac–Mörse potential was introduced and its bound-state spectrum and spinor wavefunctions were obtained [14]. Together with its two well established partners, the Dirac–Coulomb and Dirac-oscillator [15] potentials, they complete one relativistic class. In this paper, we continue these efforts by introducing the relativistic extension of yet another class of shape-invariant potentials that includes 'Dirac–Rosen–Mörse', 'Dirac–Eckart', 'Dirac–Pöschl–Teller' and 'Dirac–Scarf' potentials. We obtain their relativistic bound-state spectra and spinor wavefunctions. This is accomplished by following the same procedure that was used in [14] for the introduction and solution of the Dirac–Mörse problem.

We start by setting up the physical problem of a charged spinor in a spherically symmetric four-component electromagnetic potential. Gauge invariance and spherical symmetry of the electrostatic potential are used to arrive at the radial Dirac equation. Afterwards, we apply a unitary transformation to the Dirac equation such that the resulting second-order differential equation becomes Schrödinger-like, so that comparison with exactly solvable nonrelativistic problems is transparent. Thus, the resulting simple correspondence map among parameters of the two problems gives the sought after bound-state spectrum and wavefunction.

In atomic units ($m = e = \hbar = 1$) and taking the speed of light $c = \alpha^{-1}$, the Hamiltonian for a Dirac spinor in a four-component electromagnetic potential, $(A_0, \vec{A})$, reads

$$H = \begin{pmatrix} 1 + \alpha A_0 & -\mathrm{i}\alpha\vec{\sigma}\cdot\vec{\nabla} + \alpha\vec{\sigma}\cdot\vec{A} \\ -\mathrm{i}\alpha\vec{\sigma}\cdot\vec{\nabla} + \alpha\vec{\sigma}\cdot\vec{A} & -1 + \alpha A_0 \end{pmatrix}$$

where α is the fine-structure constant and $\vec{\sigma}$ are the three 2×2 Pauli spin matrices. In quantum electrodynamics (the theory of interaction of charged particles with the electromagnetic field), local gauge symmetry implies invariance under the transformation

$$(A_0, \vec{A}) \rightarrow (A_0, \vec{A}) + (\alpha\partial\Lambda/\partial t, \vec{\nabla}\Lambda)$$

where $\Lambda(t, \vec{r})$ is a real spacetime function. That is, adding a four-dimensional gradient of the gauge field $\Lambda(t, \vec{r})$ to the electromagnetic potential will not alter the physical content of the theory. In the laboratory frame, gauge invariance implies that the general form of the electromagnetic potential for *static* charge distribution with *spherical* symmetry is

$$(A_0, \vec{A}) = (\alpha V(r), \vec{0}) + (0, \vec{\nabla}\Lambda(r)) \equiv (\alpha V(r), \hat{r}W(r))$$

where $V(r)$ is the electrostatic potential function and $\hat{r}$ is the radial unit vector. Obviously, $W(r)$ is a gauge field that does not contribute to the magnetic field. However, fixing this gauge degree of freedom by taking $W = 0$ is not the best choice. An alternative and proper 'gauge fixing condition', which is much more fruitful, will be imposed as a constraint in equation (4) below. With this electromagnetic potential, the Dirac equation reduces to the following two-component radial differential equation:

$$\begin{pmatrix} 1 + \alpha^2 V(r) & \alpha\left(\frac{\kappa}{r} + W(r) - \frac{\mathrm{d}}{\mathrm{d}r}\right) \\ \alpha\left(\frac{\kappa}{r} + W(r) + \frac{\mathrm{d}}{\mathrm{d}r}\right) & -1 + \alpha^2 V(r) \end{pmatrix}\begin{pmatrix} g(r) \\ f(r) \end{pmatrix} = \varepsilon\begin{pmatrix} g(r) \\ f(r) \end{pmatrix} \tag{1}$$

where ε is the relativistic energy and κ is the spin–orbit coupling parameter defined as $\kappa = \pm(j + \frac{1}{2})$ for $l = j \pm \frac{1}{2}$. Equation (1) gives two coupled first-order differential equations for the two radial spinor components. By eliminating the lower component we obtain a second-order differential equation for the upper. The resulting equation may turn out to be not Schrödinger-like, i.e. it may contain first-order derivatives. We apply a general local unitary transformation that eliminates the first-order derivative as follows:

$$r = q(x) \qquad \text{and} \qquad \begin{pmatrix} g(r) \\ f(r) \end{pmatrix} = \begin{pmatrix} \cos(\rho(x)) & \sin(\rho(x)) \\ -\sin(\rho(x)) & \cos(\rho(x)) \end{pmatrix}\begin{pmatrix} \phi(x) \\ \theta(x) \end{pmatrix}. \tag{2}$$

The stated requirement gives the following constraint:

$$\frac{\mathrm{d}q}{\mathrm{d}x}\left[-\alpha^2 V + \cos(2\rho) + \alpha\sin(2\rho)(W + \kappa/q) + \alpha\frac{\mathrm{d}\rho/\mathrm{d}x}{\mathrm{d}q/\mathrm{d}x} + \varepsilon\right] = \text{constant} \equiv \eta \neq 0. \tag{3}$$

This transformation and the resulting constraint are the relativistic analogue of point canonical transformation in nonrelativistic quantum mechanics [7, 9–13]. In this paper, we consider the case of global unitary transformation defined by $q(x) = x$ and $\mathrm{d}\rho/\mathrm{d}x = 0$. Substituting these in the constraint equation (3) yields

$$\begin{aligned} V(r) &= \frac{S}{\alpha}[W(r) + \kappa/r] \\ \eta &= C + \varepsilon \end{aligned} \tag{4}$$

where $S \equiv \sin(2\rho)$ and $C \equiv \cos(2\rho)$. The first relation in (4) is the gauge fixing condition for the electromagnetic potential. The transformation defined above subject to the constraint maps the radial Dirac equation (1) into the following:

$$\begin{pmatrix} C + 2\alpha^2 V & \alpha\left(-\frac{S}{\alpha} + \frac{\alpha C}{S}V - \frac{\mathrm{d}}{\mathrm{d}r}\right) \\ \alpha\left(-\frac{S}{\alpha} + \frac{\alpha C}{S}V + \frac{\mathrm{d}}{\mathrm{d}r}\right) & -C \end{pmatrix}\begin{pmatrix}\phi(r)\\ \theta(r)\end{pmatrix} = \varepsilon\begin{pmatrix}\phi(r)\\ \theta(r)\end{pmatrix}$$

which in turn gives an equation for the lower spinor component in terms of the upper:

$$\theta(r) = \frac{\alpha}{C+\varepsilon}\left[-\frac{S}{\alpha} + \frac{\alpha C}{S}V + \frac{\mathrm{d}}{\mathrm{d}r}\right]\phi(r) \tag{5}$$

resulting in the following Schrödinger-like second-order differential equation for the upper component:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\alpha^2}{T^2}V^2 + 2\varepsilon V - \frac{\alpha}{T}\frac{\mathrm{d}V}{\mathrm{d}r} - \frac{\varepsilon^2-1}{\alpha^2}\right]\phi(r) = 0 \tag{6}$$

where $T \equiv S/C = \tan(2\rho)$.

Nonrelativistic shape-invariant potentials can be divided in two classes based on the form of their eigenfunctions. In the first class, which includes the Coulomb, oscillator and Mörse potentials, the wavefunctions are written in terms of the confluent hypergeometric functions. The relativistic extension of this class has already been established [14, 15]. In the second class, which is of interest to our present investigation, the wavefunctions are written in terms of the hypergeometric functions. This class includes Rosen–Mörse, Eckart, Pöschl–Teller and Scarf potentials. The algebraic expressions of these potentials and their properties are given in [3–5, 7, 8] and references therein. Specifically, we shall consider the hyperbolic rather than the trigonometric form of these potentials. Therefore, in our attempt to search for the relativistic extension of these potentials we shall consider expressions for $V(r)$ or $W(r)$ which are simple linear combinations of $\sinh(r)$, $\mathrm{sech}(r)$, $\tanh(r)$ etc such that the nonrelativistic potentials are reproduced in the limit. Our use of the terms 'simple' and 'linear' in the previous statement is due to the fact that these are dominant features of the relativistic theory. As examples: (1) the Dirac equation is linear in the derivative whereas the Schrödinger equation is quadratic; (2) the Dirac-oscillator potential [15] is linear in the coordinate while the oscillator potential is quadratic; (3) the Dirac–Mörse potential [14] is linear in the exponential (i.e. of the form e^{-x}) whereas the nonrelativistic Mörse potential is of the form $(1-\mathrm{e}^{-x})^2$.

Now, let us consider the case where the potential function $V(r) = D\tanh(\lambda r)$ with D and λ being real parameters. Equation (6) gives the following second-order differential equation for the upper spinor component:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\alpha D}{T}\left(\frac{\alpha D}{T} + \lambda\right)\frac{1}{\cosh^2(\lambda r)} + 2\varepsilon D\tanh(\lambda r) + \left(\frac{\alpha D}{T}\right)^2 - \frac{\varepsilon^2-1}{\alpha^2}\right]\phi(r) = 0.$$

We compare this with the Schrödinger equation for the S-wave Rosen–Mörse I potential [7]

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - A(A+\lambda)\frac{1}{\cosh^2(\lambda r)} + 2B\tanh(\lambda r) + A^2 - 2E\right]\phi(r) = 0 \tag{7}$$

where A, B and λ are real constant parameters with $\lambda A > 0$, and E is the nonrelativistic energy. The comparison gives the following correspondence between nonrelativistic and relativistic parameters:

$$\begin{aligned} A &= \alpha D/T \\ B &= D\varepsilon \\ E &= (\varepsilon^2-1)/2\alpha^2. \end{aligned} \tag{8}$$

The well known nonrelativistic bound-state spectrum of equation (7) is

$$E_n = -\frac{\lambda^2}{2}(A/\lambda - n)^2 - \frac{(B/\lambda)^2}{2(A/\lambda - n)^2} + \frac{A^2}{2} \qquad n = 0, 1, \ldots, n_{\max} < A/\lambda. \tag{9}$$

The substitution formulae in (8) give the following spectrum for this relativistic 'Dirac–Rosen–Mörse I' potential:

$$\varepsilon_n = \left\{ \left[1 + (\alpha^2 D/T)^2 - \frac{\alpha^2\lambda^2}{(\alpha D/\lambda T - n)^2} \right] \Big/ \left[1 + \left(\frac{\alpha D/\lambda}{\alpha D/\lambda T - n} \right)^2 \right] \right\}^{1/2}$$

where $n = 0, 1, 2, \ldots, n_{\max}$ and $n_{\max}$ is the smallest integer satisfying

$$\left| n_{\max} - \frac{\alpha D}{\lambda T} \right| > \frac{1}{\sqrt{(\alpha D/\lambda T)^2 + (\alpha\lambda)^{-2}}}.$$

Taking the nonrelativistic limit of this spectrum with

$$\alpha \to 0$$
$$\varepsilon_n \approx 1 + \alpha^2 E_n$$
$$T \approx \alpha\tau$$

reproduces the nonrelativistic spectrum [9] with $\tau = D/A$. The bound-state wavefunction of the nonrelativistic problem [7] is mapped, using (8), into the following upper-spinor-component wavefunction:

$$\phi_n(r) = R_n(1 - z)^{(\beta - n + \gamma_n)/2}(1 + z)^{(\beta - n - \gamma_n)/2} P_n^{(\beta - n + \gamma_n, \beta - n - \gamma_n)}(z)$$

where $P_n^{(\mu,\nu)}(z)$ is the Jacobi polynomial [16], R_n is the normalization constant and

$$z = \tanh(\lambda r)$$
$$\beta = \alpha D/\lambda T$$
$$\gamma_n = \frac{D\varepsilon_n/\lambda^2}{\beta - n}.$$

Equation (5) gives the lower spinor component in terms of the upper as

$$\theta_n(r) = \frac{\alpha\lambda}{\varepsilon_n + C} \left[-\frac{S}{\alpha\lambda} + \beta z + (1 - z^2)\frac{\mathrm{d}}{\mathrm{d}z} \right] \phi_n(r).$$

Using the differential and recursion properties of the Jacobi polynomials [16], we can write this explicitly as

$$\theta_n(r) = \frac{\alpha\lambda/\beta}{\varepsilon_n + C} R_n(1 - z)^{\mu/2}(1 + z)^{\nu/2}[-(D/\lambda^2)(\varepsilon_n + C)P_n^{(\mu,\nu)}(z) + (\beta^2 - \gamma_n^2)P_{n-1}^{(\mu,\nu)}(z)]$$

where $\mu = \beta - n + \gamma_n$ and $\nu = \beta - n - \gamma_n$.

If we now take the alternative choice of potential, $V(r) = -D \coth(\lambda r)$, and go through the same steps as above we arrive at the relativistic extension of Eckart potential [3, 7]. The bound-state spectrum and spinor wavefunction for this relativistic 'Dirac–Eckart' potential are listed in the table.

To obtain the relativistic extension of the other potentials in this class we consider the case $V = 0$, which is equivalent to the identity transformation (i.e. $\rho = 0$) combined with the constraint (4). Thus, Dirac equation (1) now reads

$$\begin{pmatrix} 1 & \alpha\left(W + \frac{\kappa}{r} - \frac{\mathrm{d}}{\mathrm{d}r}\right) \\ \alpha\left(W + \frac{\kappa}{r} + \frac{\mathrm{d}}{\mathrm{d}r}\right) & -1 \end{pmatrix} \begin{pmatrix} \phi(r) \\ \theta(r) \end{pmatrix} = \varepsilon \begin{pmatrix} \phi(r) \\ \theta(r) \end{pmatrix}.$$

This gives the following equation for the lower spinor component in terms of the upper:

$$\theta(r) = \frac{\alpha}{1+\varepsilon}\left(W + \frac{\kappa}{r} + \frac{\mathrm{d}}{\mathrm{d}r}\right)\phi(r). \tag{10}$$

Meanwhile, the upper component solves the following Schrödinger-like second-order differential equation:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\kappa(\kappa+1)}{r^2} + W^2 - \frac{\mathrm{d}W}{\mathrm{d}r} + 2\kappa\frac{W}{r} - \frac{\varepsilon^2-1}{\alpha^2}\right]\phi(r) = 0. \tag{11}$$

Now, all nonrelativistic potentials in this class are solvable only for the S-wave problem (i.e. $l = 0$), thus we restrict our analysis to the case where $\kappa = 0$. We start by considering $W(r) = F\coth(\lambda r) - G\mathrm{csch}(\lambda r)$ with F, G and λ being real constant parameters and $\lambda F > 0$. With this potential function and $\kappa = 0$, equation (11) gives the following second-order differential equation for the upper spinor component:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{F^2+G^2+\lambda F}{\sinh^2(\lambda r)} - G\,(2F+\lambda)\,\frac{\cosh(\lambda r)}{\sinh^2(\lambda r)} + F^2 - \frac{\varepsilon^2-1}{\alpha^2}\right]\phi(r) = 0.$$

Comparing this with the Schrödinger equation for the S-wave Rosen–Mörse II potential [3, 7]

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{A^2+B^2+\lambda A}{\sinh^2(\lambda r)} - B\,(2A+\lambda)\,\frac{\cosh(\lambda r)}{\sinh^2(\lambda r)} + A^2 - 2E\right]\phi(r) = 0 \tag{12}$$

gives the following correspondence between nonrelativistic and relativistic parameters:

$$\begin{aligned} A &= F \\ B &= G \\ E &= (\varepsilon^2-1)/2\alpha^2. \end{aligned} \tag{13}$$

The well known nonrelativistic bound-state spectrum of equation (12) is

$$E_n = -\frac{\lambda^2}{2}(A/\lambda - n)^2 + \frac{A^2}{2} \qquad n = 0, 1, \ldots, n_{\max} < A/\lambda. \tag{14}$$

The substitution (13) results in the following relativistic spectrum for this 'Dirac–Rosen–Mörse II' potential:

$$\varepsilon_n = \pm\sqrt{1 + \alpha^2F^2 - \alpha^2\lambda^2(F/\lambda - n)^2} \tag{15}$$

where $n = 0, 1, 2, \ldots, n_{\max}$ and $n_{\max}$ is the largest integer satisfying

$$|n_{\max} - F/\lambda| < \sqrt{(F/\lambda)^2 + (\alpha\lambda)^{-2}}.$$

It is obvious that the nonrelativistic limit ($\alpha \to 0$) of (15) reproduces the spectrum in (14). The bound-state wavefunction of the nonrelativistic problem [3, 7] is transformed, using (13), into the following upper-spinor-component wavefunction:

$$\phi_n(r) = R_n(z-1)^{(\gamma-\beta)/2}(z+1)^{-(\gamma+\beta)/2}P_n^{(\gamma-\beta-1/2,-\gamma-\beta-1/2)}(z)$$

where

$$\begin{aligned} z &= \cosh(\lambda r) \\ \beta &= F/\lambda \\ \gamma &= G/\lambda. \end{aligned}$$

Equation (10) gives the lower spinor component in terms of the upper as

$$\theta_n(r) = -\frac{\alpha\lambda}{\varepsilon_n+1}(z^2-1)^{-1/2}\left[\gamma - \beta z + (1-z^2)\frac{\mathrm{d}}{\mathrm{d}z}\right]\phi_n(r).$$

Table 1. Potential functions $V(r)$ and $W(r)$, transformation angle ρ and bound-state spectrum ε_n for the five potentials. The table is continued to show explicitly the two-component radial spinor wavefunctions $\phi_n(r)$ and $\theta_n(r)$ for each potential.

	$V(r)$	$W(r)$	$\tan(2\rho)$	ε_n
Dirac–Rosen–Mörse I	$D\tanh(\lambda r)$	$(\alpha D/S)\tanh(\lambda r)-\kappa/r$	$\alpha D/A$	$\left\{\left[1+\left(\alpha^2 D/T\right)^2-\frac{\alpha^2\lambda^2}{(\alpha D/\lambda T-n)^2}\right]\Big/\left[1+\left(\frac{\alpha D/\lambda}{\alpha D/\lambda T-n}\right)^2\right]\right\}^{1/2}$
Dirac–Eckart	$-D\coth(\lambda r)$	$-(\alpha D/S)\coth(\lambda r)-\kappa/r$	$\alpha D/A$	$\left\{\left[1+\left(\alpha^2 D/T\right)^2-\frac{\alpha^2\lambda^2}{(\alpha D/\lambda T+n)^2}\right]\Big/\left[1+\left(\frac{\alpha D/\lambda}{\alpha D/\lambda T+n}\right)^2\right]\right\}^{1/2}$
Dirac–Rosen–Mörse II	0	$F\coth(\lambda r)-G\mathrm{csch}(\lambda r)$	0	$\sqrt{1+\alpha^2F^2-\alpha^2\lambda^2\,(F/\lambda-n)^2}$
Dirac–Scarf	0	$F\tanh(\lambda r)+G\mathrm{sech}(\lambda r)$	0	$\sqrt{1+\alpha^2F^2-\alpha^2\lambda^2\,(F/\lambda-n)^2}$
Dirac–Pöschl–Teller	0	$F\tanh(\lambda r)-G\coth(\lambda r)$	0	$\sqrt{1+\alpha^2(G-F)^2-\alpha^2\lambda^2[(G-F)/\lambda+2n]^2}$

	$\phi_n(r)$	$\theta_n(r)$
Dirac–Rosen–Mörse I	$\phi_n(r)=R_n(1-z)^{\mu/2}(1+z)^{\nu/2}P_n^{(\mu,\nu)}(z)$ $z=\tanh(\lambda r),\ \mu=\beta-n+\gamma_n,\ \nu=\beta-n-\gamma_n$ $\beta=\alpha D/\lambda T,\ \gamma_n=\left(\varepsilon_n D/\lambda^2\right)(\beta-n)^{-1}$	$\theta_n(r)=\frac{\alpha\lambda/\beta}{\varepsilon_n+C}R_n(1-z)^{\mu/2}(1+z)^{\nu/2}$ $\times\left[-\left(D/\lambda^2\right)(\varepsilon_n+C)\,P_n^{(\mu,\nu)}(z)+\left(\beta^2-\gamma_n^2\right)P_{n-1}^{(\mu,\nu)}(z)\right]$
Dirac–Eckart	$\phi_n(r)=R_n(z-1)^{\mu/2}(z+1)^{\nu/2}P_n^{(\mu,\nu)}(z)$ $z=\coth(\lambda r),\ \mu=-\beta-n+\gamma_n,\ \nu=-\beta-n-\gamma_n$ $\beta=\alpha D/\lambda T,\ \gamma_n=\left(\varepsilon_n D/\lambda^2\right)(\beta+n)^{-1}$	$\theta_n(r)=\frac{\alpha\lambda/\beta}{\varepsilon_n+C}R_n(z-1)^{\mu/2}(z+1)^{\nu/2}$ $\times\left[-\left(D/\lambda^2\right)(\varepsilon_n+C)\,P_n^{(\mu,\nu)}(z)+\left(\gamma_n^2-\beta^2\right)P_{n-1}^{(\mu,\nu)}(z)\right]$
Dirac–Rosen–Mörse II	$\phi_n(r)=R_n(z-1)^{(\mu+1/2)/2}(z+1)^{(\nu+1/2)/2}P_n^{(\mu,\nu)}(z)$ $z=\cosh(\lambda r),\ \beta=F/\lambda,\ \gamma=G/\lambda$ $\mu=-\beta-1/2+\gamma,\ \nu=-\beta-1/2-\gamma$	$\theta_n(r)=\frac{\alpha\lambda}{\varepsilon_n+1}R_n(z-1)^{\frac{\mu-1/2}{2}}(z+1)^{\frac{\nu-1/2}{2}}\left\{n\left(z+\frac{\gamma}{\beta-n+1/2}\right)P_n^{(\mu,\nu)}(z)\right.$ $\left.+\left[\frac{(\beta-n+1/2)^2-\gamma^2}{\beta-n+1/2}\right]P_{n-1}^{(\mu,\nu)}(z)\right\}$
Dirac–Scarf	$\phi_n(r)=R_n(1+z^2)^{-\beta/2}\mathrm{e}^{-\gamma\tan^{-1}(z)}P_n^{(\mu,\nu)}(\mathrm{i}z)$ $z=\sinh(\lambda r),\ \beta=F/\lambda,\ \gamma=G/\lambda$ $\mu=-\beta-1/2-\mathrm{i}\gamma,\ \nu=-\beta-1/2+\mathrm{i}\gamma$	$\theta_n(r)=\frac{\alpha\lambda}{\varepsilon_n+1}R_n\left(1+z^2\right)^{-\frac{\beta+1}{2}}\mathrm{e}^{-\gamma\tan^{-1}(z)}\left\{n\left(z-\frac{\gamma}{\beta-n+1/2}\right)P_n^{(\mu,\nu)}(\mathrm{i}z)\right.$ $\left.-\mathrm{i}\left[\frac{(\beta-n+1/2)^2+\gamma^2}{\beta-n+1/2}\right]P_{n-1}^{(\mu,\nu)}(\mathrm{i}z)\right\}$
Dirac–Pöschl–Teller	$\phi_n(r)=R_n(1-z)^{\beta/2}(1+z)^{-\gamma/2}P_n^{(\mu,\nu)}(z)$ $z=\cosh(2\lambda r),\ \beta=F/\lambda,\ \gamma=G/\lambda$ $\mu=\beta-1/2,\ \nu=-\gamma-1/2$	$\theta_n(r)=\frac{2\alpha\lambda}{\varepsilon_n+1}R_n(1-z)^{\frac{\beta-1}{2}}(1+z)^{-\frac{\gamma+1}{2}}\left\{n\left(z-\frac{\beta+\gamma}{\beta-\gamma+2n-1}\right)P_n^{(\mu,\nu)}(z)\right.$ $\left.-2\left[\frac{(\beta+n-1/2)(-\gamma+n-1/2)}{\beta-\gamma+2n-1}\right]P_{n-1}^{(\mu,\nu)}(z)\right\}$

Again, using the differential and recursion properties of the Jacobi polynomials [16], we can write this explicitly as

$$\theta_n(r) = \frac{\alpha\lambda}{\varepsilon_n + 1} R_n (z-1)^{(\mu-1/2)/2}(z+1)^{(\nu-1/2)/2}\left\{ n\left(z + \frac{\gamma}{\beta - n + 1/2}\right) P_n^{(\mu,\nu)}(z) + \left[\frac{(\beta - n + 1/2)^2 - \gamma^2}{\beta - n + 1/2}\right] P_{n-1}^{(\mu,\nu)}(z)\right\}$$

where $\mu = \gamma - \beta - 1/2$ and $\nu = -\gamma - \beta - 1/2$.

Taking the alternative choice $W(r) = F\tanh(\lambda r) + G\,\mathrm{sech}(\lambda r)$ and going through the same steps as above we arrive at the relativistic extension of the Scarf potential [3, 17]. The bound-state spectrum and spinor wavefunction for this 'Dirac–Scarf' potential are listed in table 1. The table also lists the 'Dirac–Pöschl–Teller' potential $W(r) = F\tanh(\lambda r) - G\coth(\lambda r)$, which, in the nonrelativistic limit, reproduces the usual Pöschl–Teller potential [7, 18, 19].

Finally, it is worth noting that it would be of prime relevance, as a future development, to find the general transformations $q(x)$ and $\rho(x)$ in (2) that map any one of these relativistic potentials into other members of the class. Moreover, it might be possible that an exhaustive study of such transformations may bring about new relativistic potentials that enlarge the class. A similar treatment is called for concerning the other class of relativistic potentials that includes Dirac–Coulomb, Dirac-oscillator and Dirac–Mörse potentials.

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